1 (a)

$f(x) = x^3 + x \rightarrow f(1) = 2$

$f'(x) = 3x^2 + 1 \rightarrow f'(1) = 4$

$f''(x) = 6x \rightarrow f''(1) = 6$

$f'''(x) = 6 \rightarrow f'''(1) = 6$

$f''''(x) = 0 \rightarrow f''''(1) = 0$

$f'''''(x) = 0 \rightarrow f'''''(1) = 0$

$\vdots \qquad \vdots$

rest are all zero

Thus,

$$f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots$$

$$x^3 + x = 2 + 4(x-1) + \frac{6}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3$$

$$x^3 + x = 2 + 4(x-1) + 3(x-1)^2 + (x-1)^3$$

Since the above is a finite polynomial and the sum converges for $-\infty < x < \infty$ the radius of convergence is $r = \infty$

$\boxed{\text{① (b)}}$ $f(x) = x$

$f(x) = x \longrightarrow f(-2) = -2$

$f'(x) = 1 \longrightarrow f'(-2) = 1$

$f''(x) = 0 \longrightarrow f''(-2) = 0$

$f'''(x) = 0 \longrightarrow f'''(-2) = 0$

$\vdots \qquad\qquad \vdots$

rest are all zero

So,

$$f(x) = f(-2) + f'(-2)\underbrace{(x+2)}_{(x-(-2))} + \frac{f''(-2)}{2!}\underbrace{(x+2)^2}_{(x-(-2))^2} + \cdots$$

$$\boxed{x = -2 + (x-2)}$$

This sum converges for $-\infty < x < \infty$ since its a finite polynomial. So, the radius of convergence is $r = \infty$.

1(c)

$f(x) = x^2 \longrightarrow f(1) = 1$

$f'(x) = 2x \longrightarrow f'(1) = 2$

$f''(x) = 2 \longrightarrow f''(1) = 2$

$f'''(x) = 0 \longrightarrow f'''(1) = 0$

$f''''(x) = 0 \longrightarrow f''''(1) = 0$

$\vdots \qquad \vdots$

rest are all zero

So,

$$x^2 = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \dots$$

$$\boxed{x^2 = 1 + 2(x-1) + (x-1)^2}$$

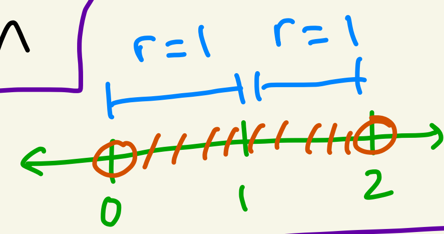
This has radius of convergence $r = \infty$ and converges for $-\infty < x < \infty$ since its a finite polynomial

① (d) Find a power series expansion for $f(x) = \frac{1}{x}$ at $x_0 = 1$.

If we only look at $x > 0$, then

$$\frac{1}{x} = \frac{d}{dx} \ln(x)$$

When $0 < x < 2$



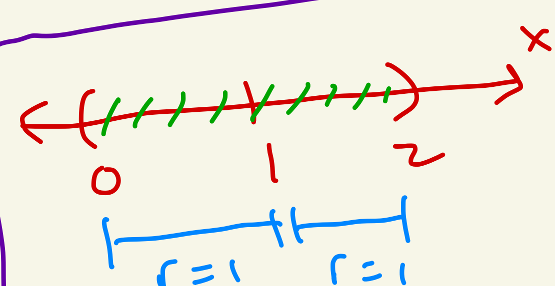
$$= \frac{d}{dx}\left[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots \right]$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$$

So,

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

Since we differentiated the $\ln(x)$ series with radius of convergence $r=1$, this resulting series for $\frac{1}{x}$ will also have radius of convergence $r=1$.



$\boxed{①(e)}$

We know that

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

for $-\infty < x < \infty$,
with radius of convergence $r = \infty$.

Plug x^2 into the formula to get:

$$e^{x^2} = 1 + x^2 + \frac{1}{2!}(x^2)^2 + \frac{1}{3!}(x^2)^3 + \frac{1}{4!}(x^2)^4 + \frac{1}{5!}(x^2)^5 + \cdots$$

$$= 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + \frac{1}{5!}x^{10} + \cdots$$

for $-\infty < x < \infty$,
with radius of convergence $r = \infty$.

① (f)

$$f(x) = \frac{-x}{1-x^2} = -x\left(\frac{1}{1-x^2}\right)$$

$$= -x\left(1 + x^2 + (x^2)^2 + (x^2)^3 + (x^2)^4 + \dots\right)$$

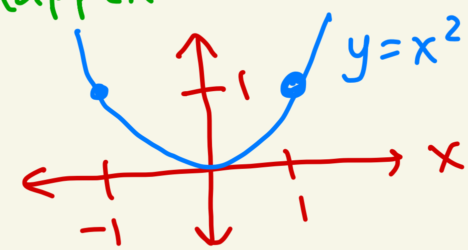
Geometric sum:

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \dots$$

$-1 < u < 1$ or $|u| < 1$

need $\overbrace{-1 < x^2 < 1}^{-1<u<1}$ to use the geometric sum formula

This happens when $-1 < x < 1$

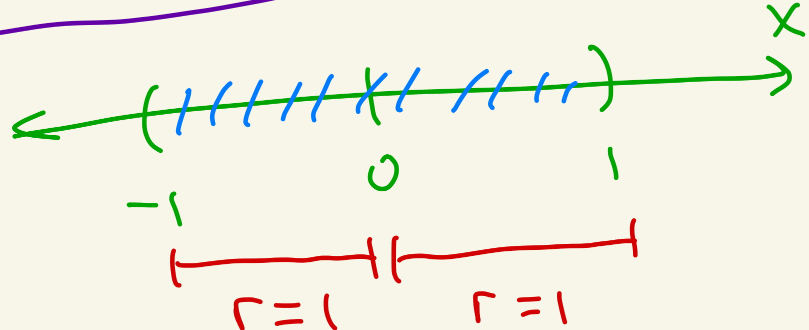


So,

$$\frac{-x}{1-x^2} = -x\left(1 + x^2 + x^4 + x^6 + x^8 + \dots\right)$$

$$= -x - x^3 - x^5 - x^7 - x^9 - \dots$$

has radius of convergence $r = 1$ since it converges when $-1 < x < 1$.



$\boxed{②\ (a)}$ From class,

$$\sin(x) = \underbrace{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7}_{\text{first 5 terms}} + \frac{1}{9!}x^9 - \ldots$$

for $-\infty < x < \infty$.

Thus an estimate for $\sin(0.1)$ is

$$0.1 - \frac{1}{3!}(0.1)^3 + \frac{1}{5!}(0.1)^5 - \frac{1}{7!}(0.1)^7$$

$$= 0.1 - \frac{1}{6}(0.001) + \frac{1}{120}(0.00001) - \frac{1}{5040}(0.0000001)$$

$$= \boxed{0.0998334}$$

$\boxed{(b)}$ My calculator says that

$$\sin(0.1) \approx \boxed{0.0998334166468 28\ldots}$$

We were very close!

3 (a) From class,

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

$$= \underbrace{(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4}_{\text{first 4 terms}} + \frac{1}{5}(x-1)^5 - \ldots,$$

This is valid for $0 < x < 2$.

Thus an estimate for $\ln(1.1)$ is

$$(1.1-1) - \frac{1}{2}(1.1-1)^2 + \frac{1}{3}(1.1-1)^3 - \frac{1}{4}(1.1-1)^4$$

$$= 0.1 - \frac{0.1^2}{2} + \frac{0.1^3}{3} - \frac{0.1^4}{4}$$

$$= 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4}$$

$$= \boxed{0.0953083}$$

(b) My calculator says that

$$\boxed{\ln(1.1) \approx 0.09531018\ldots}$$

That's pretty close, 4 decimal places of accuracy!